Imperial College London

Lecture 7

Step Response & System Behaviour

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In this lecture, I will continue to consider Laplace transform, particularly for a 1st order and a 2nd order system. I will develop some insights into how these systems behave both in the time domain in response to a step input, and in the frequency domain (that is, in response to sinusoids at different frequencies).

◆ Consider what happens to the circuit shown here as the switch is closed at t = 0. We are interested in $y(t)$.

◆ Apply KVL around the loop, we get:

$$i(t)R + y(t) = x(t), but\ i = C\frac{dy}{dt}\ \ therefore$$

$$RC\frac{dy}{dt} + y = x$$
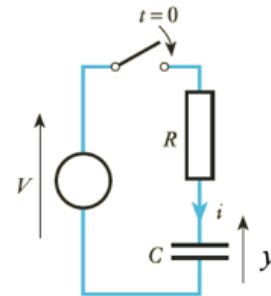
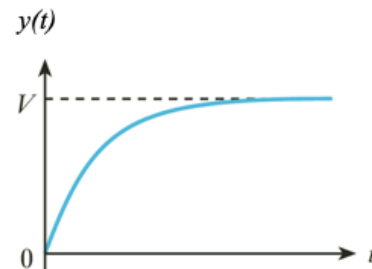◆ This is a simple first-order differential equation with constant coefficients.

◆ We can model closing the switch at t=0 as:

$$x(t) = V\ u(t)$$

◆ Then the solution of the differential equation is:

$$y(t) = V\left(1 - e^{-\frac{t}{RC}}\right)u(t)$$

◆ You should be familiar with this from Electronics 1 last year: τ = RC, the time-constant

Let us first consider a simple RC circuit, which you have learn from last year. This slide is taken from Lecture 8, slide 3. from DE1.3 last year. Here I assume that you are familiar with solving first-order differential equations from your maths lectures.

We want to solve (x(t) = V u(t)):

$$RC\frac{dy}{dt} + y = Vu(t)$$

$$\frac{dy}{dt} = \frac{Vu(t) - y}{RC} \implies \frac{dt}{RC} = \frac{dy}{Vu(t) - y}$$

Integrate both sides, we get:

$$\frac{t}{RC} = -\ln(Vu(t) - y) + A$$ where A is constant of integration

Use boundary condition: when t= 0, y = 0:

$$\frac{0}{RC} = -\ln(Vu(t) - 0) + A$$
$$\implies A = \ln(Vu(t))$$
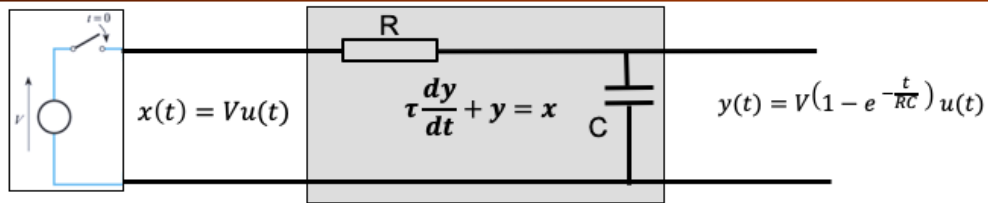
Therefore

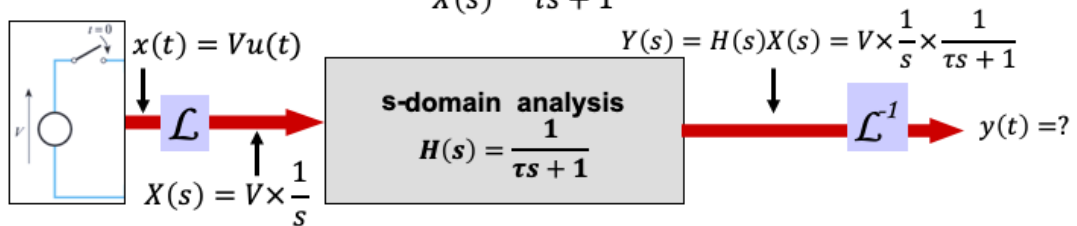$$\frac{t}{RC} = -\ln(Vu(t) - y) + \ln(Vu(t)) = ln\frac{Vu(t)}{Vu(t) - y}$$

$$\implies e^{\frac{t}{RC}} = \frac{Vu(t)}{Vu(t) - y} \implies y(t) = V\left(1 - e^{-\frac{t}{RC}}\right)u(t)$$

Modelling using Laplace Transform

In the previous slide, we use time domain analysis and differential equations. Now let us move to the Laplace s-domain, and use Transfer function to do the same analysis. The top diagram is the time domain view of things.

Let us first take the Laplace transform of the input x(t) = V u(t):

$$\mathcal{L}\{x(t)\} = X(s) = \mathcal{L}\{V \times u(t)\} = V \times \mathcal{L}\{u(t)\} = V \times \frac{1}{s}$$

Remember that, from L6 S13, we know the LT of unity step function u(t) is 1/s.

Now we take the Laplace transform of the differential equation, remembering from L6 S15 that:

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s)$$

Therefore:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{\tau s + 1} = \frac{1/\tau}{s + 1/\tau}$$

Finally, we known $Y(s) = H(s)X(s)$

Therefore:

$$Y(s) = Vs \times \frac{1}{s} \times \frac{1/\tau}{s + 1/\tau}$$

However, we are interested in y(t), not Y(s). So how do we convert from Laplace domain back to time domain? For that, we need inverse Laplace Transform.

Forward & Inverse Laplace Transform

◆ Remember: the definition of the Laplace Transform $\mathcal{L}$ is:

$$\mathcal{L}[x(t)] = X(s) = \int_0^\infty x(t)e^{-st}\,dt$$

◆ The definition of the Inverse Laplace Transform $\mathcal{L}^{-1}$ is:

$$\mathcal{L}^{-1}[X(s)] = x(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\varpi} X(s)e^{st}\,ds, \quad \omega \to \infty$$

During the lecture, I will go through the next few slides quickly because these are mostly included for completeness. We don't usually perform Laplace Transform mathematically by hand. Instead we would use a table and look the LT results up. Nevertheless, you should at least remember the definition of the forward transform.

Here is a reminder of the equation for forward Laplace transform for a causal signal is shown here. You need to remember this!

$$\mathcal{L}[x(t)] = X(s) = \int_0^\infty x(t)e^{-st}\,dt$$

The inverse Laplace transform is more complicated. It is defined below. You DO NOT need to remember this.

$$\mathcal{L}^{-1}[X(s)] = x(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\varpi} X(s)e^{st}\,ds, \quad \omega \to \infty$$

For this course (and for most practical applications), we DO NOT calculate the inverse Laplace transform by hand. Instead, we do most of the forward and inverse transformations via looking up a transform a table. I have included these formulae here just for completeness and for reference.

Laplace transform Pairs (1)

- ◆ Finding inverse Laplace transform requires integration in the complex plane – beyond scope of this course.
- ◆ So, use a Laplace transform table (analogous to the Fourier Transform table).

No.	$x(t)$	$X(s)$
* 1	$\delta(t)$	1
* 2	$u(t)$	$\dfrac{1}{s}$
3	$tu(t)$	$\dfrac{1}{s^2}$
4	$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$

L4.1 p344

The table of Laplace transform pairs (going both directions) is taken from Lathi's book. The first TWO shown here are useful, particularly for signals and systems.

The first pair is the impulse function. The LT is the constant 1.

Pair 2 is the LT of the unity step function, and we have seen in L6 S13 that this is computed to be 1/s.

Laplace transform Pairs (2)

No.	$x(t)$	$X(s)$
* 5	$e^{\lambda t}u(t)$	$\dfrac{1}{s-\lambda}$
6	$te^{\lambda t}u(t)$	$\dfrac{1}{(s-\lambda)^2}$
7	$t^n e^{\lambda t}u(t)$	$\dfrac{n!}{(s-\lambda)^{n+1}}$
* 8a	$\cos bt\, u(t)$	$\dfrac{s}{s^2+b^2}$
* 8b	$\sin bt\, u(t)$	$\dfrac{b}{s^2+b^2}$
* 9a	$e^{-at}\cos bt\, u(t)$	$\dfrac{s+a}{(s+a)^2+b^2}$
* 9b	$e^{-at}\sin bt\, u(t)$	$\dfrac{b}{(s+a)^2+b^2}$

L4.1 p344

Pair 5 here is MOST important. You will find that most systems will have terms in the form of $\dfrac{1}{s-\lambda}$ in the s-domain. The time domain equivalent of this is a causal exponential function $e^{\lambda t}u(t)$. The unity step function u(t) makes this causal, meaning that it is zero for t < 0.   The term $e^{\lambda t}$ is the general solution for most differential equations. It represents the natural response of many physical systems.

Pairs 8a and 8b are also important because they represent the LT of causal sine and cosine waveforms.

Finally, 9a and 9b represents exponential decaying, causal sine and cosine, something that occurs frequently in the physical world.

Finding Inverse Laplace Transform via partial fraction

◆ Finding inverse Laplace transform of $Y(s) = \frac{1}{s} \times \frac{1/\tau}{s + 1/\tau}$ (use partial fraction)

$$Y(s) = \frac{1}{s} \times \frac{1/\tau}{s + 1/\tau} = \frac{k_1}{s} + \frac{k_2}{s + 1/\tau}$$

◆ To find k_1 which corresponds to the term (s+0) in denominator, cover up (s+0) in Y(s), and substitute s = 0 (i.e. s+0=0) in the remaining expression:

$$k_1 = \frac{1}{s} \times \frac{1/\tau}{s + 1/\tau}\Big|_{s=0} = 1$$

◆ Similarly for k_2, cover the (s+1/τ) term, and substitute s = -1/τ, we get:

$$k_2 = \frac{1}{s} \times \frac{1/\tau}{s + 1/\tau}\Big|_{s=-1/\tau} = -1$$

◆ Therefore

$$Y(s) = \frac{1}{s} - \frac{1}{s + 1/\tau}$$

L4.1 p349

Now that we have learned about forward and inverse Laplace transform, let us find y(t) for our simple circuit assuming that the circuit input is a step function of magnitude V. To simplify things a bit, let us assume that V = 1, and

$$Y(s) = \frac{1}{s} \times \frac{1/\tau}{s + 1/\tau}$$

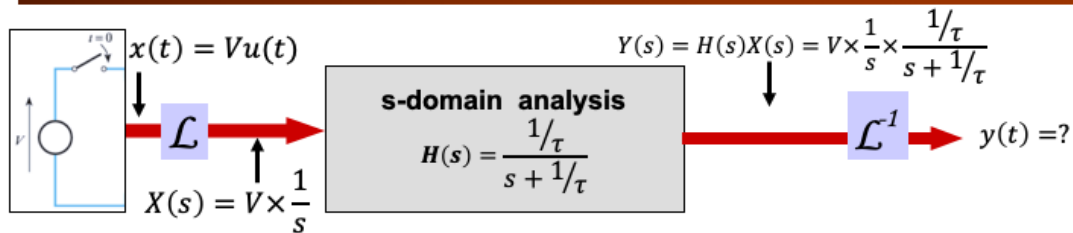How can we find the inverse Laplace transform of Y(s) to obtain y(t)?

For this, we need to:

1. Factorise the numerator and denominator of Y(s) (i.e. turn them into products of terms in the form of (s + a), where a is a constant, and s is the complex Laplace variable.

2. We then use partial fraction to turn each product terms into the form:

$$\frac{k_1}{s+a_1} + \frac{k_2}{s+a_2} + \dots + \frac{k_n}{s+a_n}$$

3. We then computer the constants k_i using the technique shown here.

4. To find k_1, which corresponds to the term 1/s, we first "cover" 1/s (i.e. remove this from the expression), and substitute s = 0 into the remaining expression.

5. To find k_2, which corresponds to the term 1/(s + 1/τ), we first "cover" 1/(s + 1/τ) (i.e. remove this from the expression), and substitute s = - 1/τ into the remaining expression. (We use the solution of (s + 1/τ) = 0 for this.)

This technique works for all partial fraction calculations.

$x(t) = Vu(t)$

$\mathcal{L}$

$X(s) = V \times \dfrac{1}{s}$

s-domain analysis

$H(s) = \dfrac{1/\tau}{s + 1/\tau}$

$Y(s) = H(s)X(s) = V \times \dfrac{1}{s} \times \dfrac{1/\tau}{s + 1/\tau}$

$\mathcal{L}^{-1}$

$y(t) = ?$

- So, we get: $Y(s) = V\left(\dfrac{1}{s} - \dfrac{1}{s + 1/\tau}\right)$

- Use Laplace Transform table, pair 5: $e^{\lambda t}u(t) \overset{\mathcal{L}}{\Longleftrightarrow} \dfrac{1}{s-\lambda}$

$$\mathcal{L}^{-1}\{Y(s)\} = V\mathcal{L}^{-1}\left\{\dfrac{1}{s} - \dfrac{1}{s + 1/\tau}\right\} = V\left(u(t) - e^{-\frac{t}{\tau}}u(t)\right) = V \times (1 - e^{-\frac{t}{\tau}}) \times u(t)$$

- Same as results from slide 2 using differential equation.

Now that we have rearrange Y(s) into partial fractions, we can perform the inverse transform easily because each term is of the form $\dfrac{1}{s - \lambda}$.

Using tranform table, Pair 5, we found that the inverse transform of each of the terms $\dfrac{1}{s - \lambda}$ get a time domain response of the form $e^{\lambda t}u(t)$!

For our circuit, we obtain the same results as before: it is a rising exponential from 0 to V with a time constant of τ.

Such exponential signal is something you have encountered as early as Lab 1 of DE1.3 last year. It is one of the most common behaviour of many systems.

Another Examples of Inverse Laplace Transform

- Finding the inverse Laplace transform of $\dfrac{2s^2-5}{(s+1)(s+2)}$.

- The partial fraction of this expression is less straight forward. If the power of numerator polynomial (M) is the same as that of denominator polynomial (N), we need to add the coefficient of the highest power in the numerator to the normal partial fraction form:

$$X(s) = 2 + \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

- Solve for k_1 and k_2 via "covering":

$$k_1 = \frac{2s^2+5}{(s+1)(s+2)}\Big|_{s=-1} = \frac{2+5}{-1+2} = 7$$

$$k_2 = \frac{2s^2+5}{(s+1)(s+2)}\Big|_{s=-2} = \frac{8+5}{-2+1} = -13$$

- Therefore $\quad X(s) = 2 + \dfrac{7}{s+1} - \dfrac{13}{s+2}$

- Using pairs 1 & 5:

$$x(t) = 2\delta(t) + (7e^{-t} - 13e^{-2t})u(t)$$

L4.1 p350

Here is yet another example to demonstrate how we might find the inverse transform of a function in s-domain. Follow this through yourself in your spare time, to make sure that you understand this.

Transfer function of a light bulb

◆ In Lab 2, we use the Bulb Box system, and it was known that the light bulb part of the system has a transfer function as shown:

◆ Therefore the light bulb itself has an exponential response with a time constant τ = 38 ms.

◆ Remember, for a 1st order system, the output step response reaches the following percentages of final value after n x τ, n=1,2,3,...:

Time =	τ	2τ	3τ	4τ
Final value	63.2%	86.5%	95%	98.2%

Now let us look at a real physical 1st order system. You were using the Bulb Box in Lab 2, which consist of the light bulb circuit (and a 2nd order electronic system before that, which we will ignore for now).

The Transfer function of the light bulb part of the system:

$$B(s) = \frac{1}{0.038s + 1}$$

From the results we got in previous slide, the step response of the light bulb is a rising exponential with a time constant of τ = 0.038.

This means that the filament in the bulb takes time to heat up, and its illumination rises exponentially with a time constant τ of 38ms!

Let me remind you that in such a system, the signal reaches 63.2% after the time = time constant (i.e. 38ms in our case). The table above shows how long it takes to reach different % of final values in the case of a 1st order system.

◆ Once you know the transfer function B(s) of a system, you can evaluate its frequency response by evaluating H(s) at s = jω:

$$B(j\omega) = B(s)\Big|_{s=j\omega}$$

◆ Therefore, for our light bulb (not including the 2nd order electronic circuit, the frequency response is:

$$B(j\omega) = \frac{1}{(1 + 0.038s)}\Big|_{s=j\omega}$$

$$|B(j\omega)| = \frac{1}{|(1+0.038j\omega)|} = \frac{1}{\sqrt{1+0.038^2\omega^2}}$$

◆ From DE1.3, you know that this is a low pass filter – gain drops with increasing frequency.

Let us compare the formal definition of Laplace and Fourier transforms for a causal signal:

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^\infty x(t)e^{-st}dt$$

$$\mathcal{F}\{x(t)\} = X(\omega) = \int_0^\infty x(t)e^{-j\omega t}dt$$

Therefore, we can find the Fourier transform of any function or signal, by substituting s = jω into the Laplace domain function or signal.

In other words, the Frequency response of a system can be computed with:

$$H(\omega) = H(s)\Big|_{s=j\omega}$$

The notation here means: evaluate H(s) by substituting s=jω into the equation.

With this, we can calculate the frequency response of the light bulb. It is effectively a lowpass filter with very low frequency cutoff frequency (i.e. start to tail off at low frequency).

Transfer Function of a 2nd order system

- Let us consider a general second order system with a transfer function of the general form:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}$$

- To simplify the problem a bit, let us assuming that b2 = b1 = 0. The above equation can be rewritten as:

$$H(s) = \frac{b_0}{s^2 + a_1 s + a_0} = K \frac{\omega_0^2}{s^2 + 2\zeta \omega_0 s + \omega_0^2}$$

- where:
  - $\omega_0 = \sqrt{a_0}$,   the resonant (or natural) frequency in rad/sec
  - $\zeta = \frac{a_1}{2\sqrt{a_0}}$,   the damping factor (no unit) (pronounced as zeta)
  - $K = \frac{b_0}{a_0}$,   gain of the system

Now let us move to a 2nd order system. A general 2nd order system transfer function takes the form:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}$$

Let us simplify this by assuming that b_2 and b_1 are both zero. This gives us the transfer function found in many systems, including the one we use for Lab 2, of:

$$H(s) = \frac{b_0}{s^2 + a_1 s + a_0} = K \frac{\omega_0^2}{s^2 + 2\zeta \omega_0 s + \omega_0^2}$$

The reason for this rearrangement of the equation is that the new form provides parameters (i.e. values which are constants) that have physical meaning.

ω_0 is the **natural** or **resonant frequency** – the frequency that the system will tend to oscillate at.

ζ is the **damping factor**. It value is centred around 1. At 1, the system is known as critically damped (will be discussed later). If $\zeta < 1$, then the system will oscillate when "kicked" by a transient such as a step function. If $\zeta > 1$, then the system is behaving slower than it need be.

K is the **gain** at zero frequency, which is the **DC gain**.

Physical meaning of ω_0, ς, and K

- Let us take the transfer function H(s) of the 2nd order system used in Bulb Box as an example:

| x(t) | Electronic Circuit to emulate a 2nd order oscillatory system | p(t) |

$$X(s) \quad H(s) = \frac{1000}{s^2 + 5s + 1000} \quad P(s)$$

- $\omega_0 = \sqrt{a_0} = 31.62$,   the resonant frequency = 5Hz

- $\zeta = \frac{a_1}{2\sqrt{a_0}} = \frac{5}{2\sqrt{1000}} = 0.079$ the damping factor (very small, ideal = 1)

- $K = \frac{b_0}{a_0} = 1$,   gain of the system at DC or zero frequency

- Since the damping factor is very small (much smaller than 1), this system is highly oscillatory.

$$H(s) = \frac{b_0}{s^2 + a_1 s + a_0} = K \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Let us consider our Bulb Box system in Lab 2. The second order electronic circuit has a transfer function as shown.

The resonant frequency is 31.62 rad/sec, or 5 Hz. The damping factor is 0.079, which is way below 1, hence the system is very prone to oscillation. K is of the circuit is 1.

The importance of damping factor

- Let us consider the transfer function H(s) again:

$$H(s) = \frac{b_0}{s^2 + a_1 s + a_0} = K \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

- The unit step response of the system is (i.e. $x(t) = u(t), and\ X(s) = 1/s$):

$$Y(s) = \frac{1}{s}H(s) = \frac{1}{s}K\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

- We want to say something about the dynamic characteristic of this system by finding the natural frequency ω_0 and the damping factor ζ.
- To do that, we find need to find the root of the quadratic: $s^2 + 2\zeta\omega_0 s + \omega_0^2$

$$s = \frac{-2\zeta\omega_0 \pm \sqrt{(2\zeta\omega_0)^2 - 4\omega_0^2}}{2}$$
$$= -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

Why is this form of the equation important for a 2nd order system? Let us consider what happens if we apply a unity step function to it. The output of the system Y(s) is as shown.

We have seen before that to get the time domain output y(t), we need to take the inverse Laplace of this function. To do that, we need to first factorise the denominator, particularly the quadratic function: $s^2 + 2\zeta\omega_0 s + \omega_0^2$

The root of this quadratic is well-known:

$$s = \frac{-2\zeta\omega_0 \pm \sqrt{(2\zeta\omega_0)^2 - 4\omega_0^2}}{2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

Five cases of behaviour

- Depending on the value of the damping factor ζ, there are five cases of interest, each having a specific behaviour:

$$H(s) = \frac{b_0}{s^2 + a_1 s + a_0} = K \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

- Root of denominator:

$$s = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

Name	Value of ζ	Roots of s	Characteristics of "s"
Overdamped	$\zeta > 1$	$s = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$	Two real and negative roots
Critically Damped	$\zeta = 1$	$s = -\omega_0$	A single negative roots
Underdamped	$0 < \zeta < 1$	$s = -\zeta\omega_0 \pm j\omega_0\sqrt{1 - \zeta^2}$	Complex conjugate $(j = \sqrt{-1})$;
Undamped	$\zeta = 0$	$s = \pm j\omega_0$	Pure imaginary (no real part)
Exponential Growth	$\zeta < 0$	$s = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$	Roots may be complex or real, but the real part of s is always positive

This result is important. We will examine this in another lecture when we consider how to gain insights into a system via something called "poles" and "zeros" in a later lecture.

For now, remember that the root for s in the partial fraction process is given by:

$$s = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

The value s take on a property depending on the value of the damping factor ζ!

If $\zeta = 1$, the square root term = 0, and we have $s = -\omega_0$. We have a single negative root and the system has an exponential behaviour without any oscillation. The system is now CRITICALLY DAMPED - that is, while there is no oscillation, yet it approaches the final value of a step response fastest.

However, if ζ is below 1, (but above 0), like our Bulb Box system where $\zeta = 0.079$, the square root term is negative. The root is now a complex number with an imaginary part.

$$s = -\zeta\omega_0 \pm j\omega_0\sqrt{1 - \zeta^2}$$

The system is underdamped and imaginary part of the root indicates that there is oscillation in the system.

The table here shows the five different mode of behaviour in the system depending on the value of the damping factor ζ .
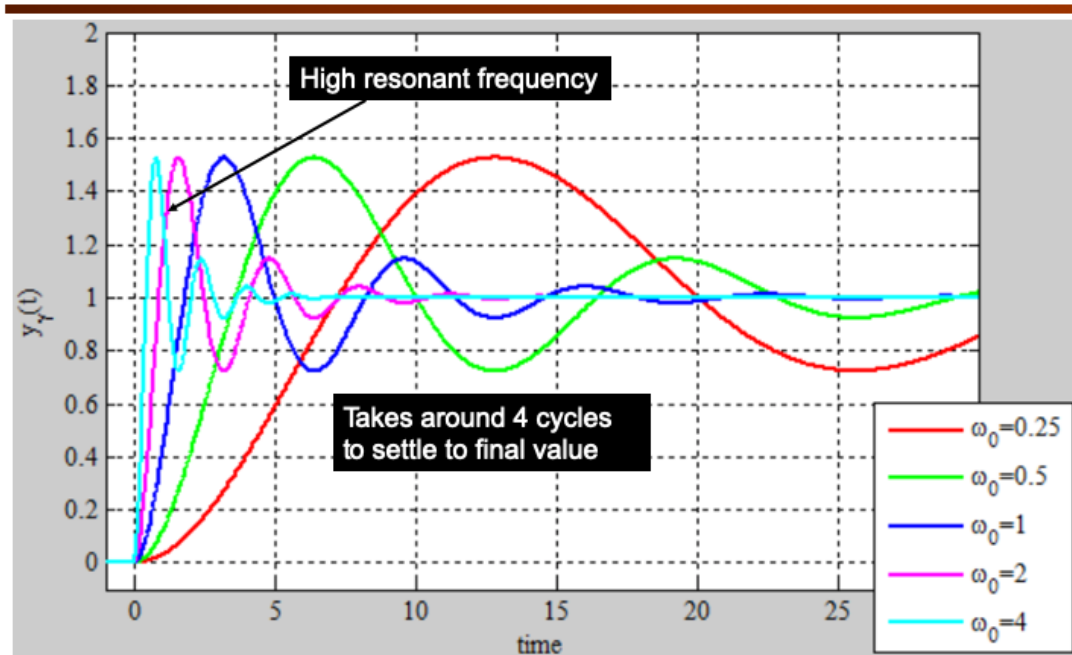
Step Response for different damping factors
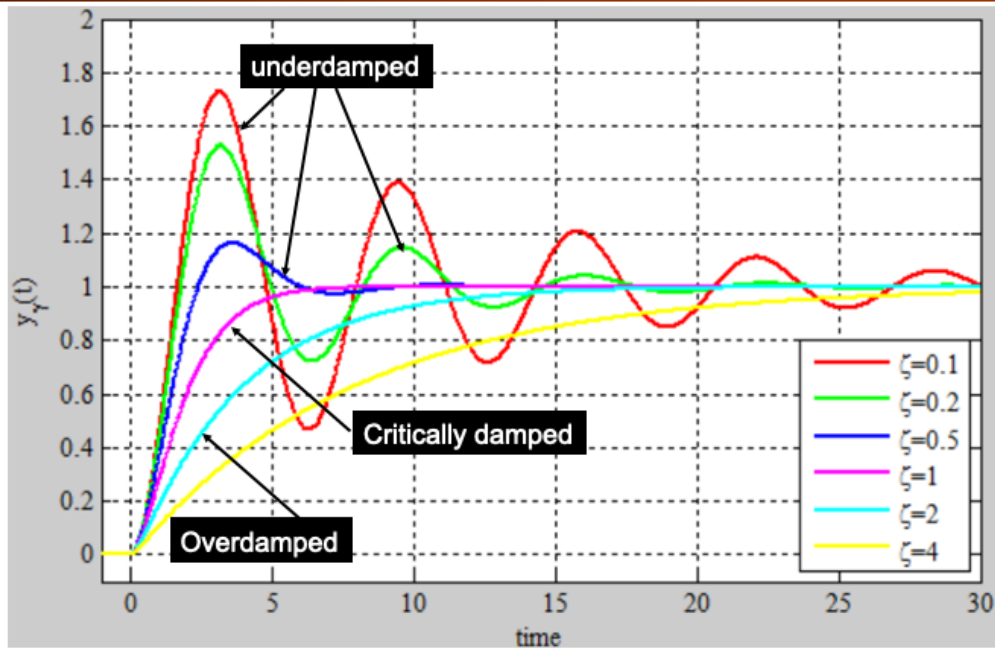
For a second order system, the unit step response is shown here for different value of ζ .
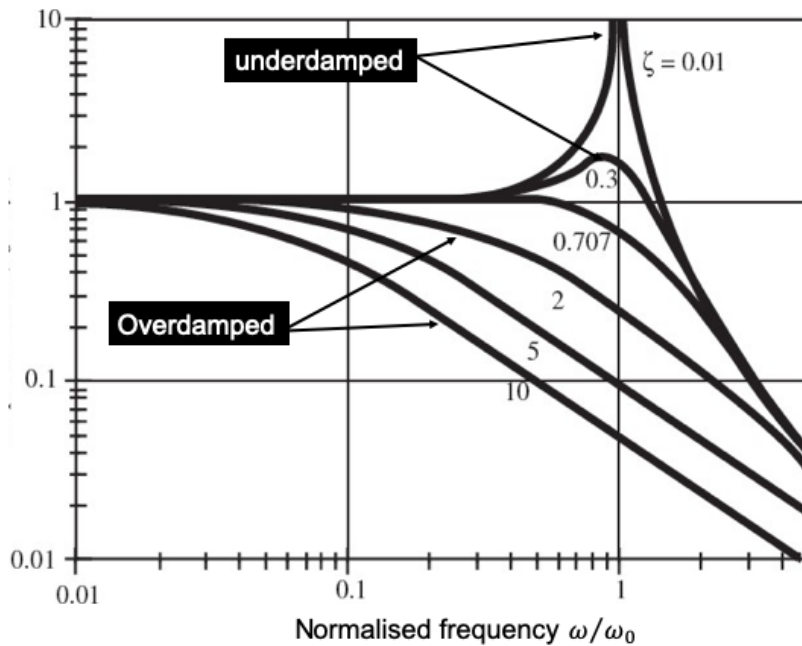
Step Response at $\omega_0, \varsigma = 0.2$

Now let us fix ζ = 0.2 and change the resonant frequency ω_0. Note that the time it takes for the signal to reach (nearly) the final value is related to the number of cycles of oscillation.

Frequency response of 2nd order system

Finally, if we substitute s = jw into the transfer function of the second order system and compute the system gain at different frequency (the same as what you did in Lab 2, exercise 2), you get the frequency response as that shown in the slide.

Here we normalise the frequency axis with the resonant frequency ω_0 and plot the system gain for different damping factor value.

Both axes are plotted in log scale. As can be seen, the system has a high gain at or around the resonant frequency, explaining why it has a tendency to oscillate at this frequency.

A video demonstrating an underdamped oscillatory system

TACOMA NARROWS
BRIDGE COLLAPSE

Length of center span	2800 ft
Width	39 ft
Depth of stiffening girders	8 ft
Start of construction	Nov. 23, 1938
Opened for traffic	July 1, 1940
Collapse of bridge	Nov. 7, 1940

This video is that of a famous Tacoma bridge in USA that collapsed. It demonstrate how a poorly damped (underdamped) system would oscillate at the resonant frequency when there is gust of wind blowing onto in.

Sudden gust of wind is like having a step function at the input of the system. If the wind is also "oscillating" at or around the resonant frequency, the input is "amplified" by the oscillatory response of the system. Our underdamped Bulb Box system is mimicking such a behavour at a resonant frequency of 5 Hz.

https://www.youtube.com/watch?v=3mclp9QmCGs

The Millennium Bridge

You must have heard of the phrase "history always repeats itself". This is a very interesting example. In 2000, London opened its newest designed pedestrian bridge linking St Paul cathedral and the Tate Modern Gallery.

Shortly after it was opened, it had to be closed. The designers did not expect the bridge would wobble side ways when pedestrians walk on it. Here is a video of what happens.

https://www.youtube.com/watch?v=y2FaOJxWqLE